13th International Conference on Applications of Statistics and Probability in Civil Engineering, ICASP13
Seoul, South Korea, May 26-30, 2019

Corotational Response Sensitivity Analysis of Space Frame Finite Elements

Alabbas Al-Aukaily
Staff Faculty Member, Department of Civil Engineering, Wasit University, Wasit, Iraq

Michael H. Scott
Professor, School of Civil and Construction Engineering, Oregon State Univ., Corvallis, OR, USA

ABSTRACT: Response sensitivity computations for nonlinear geometry are required in structural reliability, optimization, and system identification. Compared to finite difference and complex perturbation methods, the direct differentiation method (DDM) of the governing response equations offers an approach to response sensitivity computations that is both accurate and efficient. Several formulations of DDM response sensitivity have been developed for material nonlinear planar and space frame finite elements; however, the DDM has only been applied to geometrically nonlinear frame elements in two dimensions. The corotational formulation of large displacements is versatile for three dimensional simulations of frame structures because it separates geometric and material nonlinearity at the element level. However, the main difficulty of the corotational formulation for space frame elements is that finite rotations are not true vector quantities and the structural response depends on the order in which rotations are applied. Through analytic differentiation of an approximate algorithm to apply rotations, the response sensitivity equations are obtained for the corotational formulation of space frame elements and are implemented in the OpenSees finite element framework. To verify the accuracy of the derived sensitivity equations, standalone sensitivity analysis compares the proposed DDM with the finite difference method (FDM) for a space frame structure.

1. INTRODUCTION

Uncertainty quantification is an essential component to the design and assessment of engineered structures. As simulation capabilities advance, capabilities for uncertainty quantification need to keep pace. Performance-based approaches to assessment require the probability of a structural failure across limit states ranging from service level to collapse prevention.

Monte Carlo simulation (MCS) is a straightforward means of computing probabilities of failure for any structural system and external loads characterized by properties with joint probability distribution functions. Although it is widely applicable, the drawback to MCS is its computational expense, requiring at least 10,000 simulations to get an accurate estimate for a probability of failure.

The first order reliability method (FORM) offers an efficient alternative to MCS in that it requires a relatively small number of simulations to estimate a probability of failure. Whereas MCS can handle nonlinearity of the limit state, the FORM estimate of failure probability is first order approximation. The second order reliability method (SORM) makes a second order approximation in estimating the probability of failure.

FORM and SORM fall in to the wider class of gradient-based applications in structural engineering where the gradients, or first derivatives, of the structural response with respect to each uncertain parameter are required in order to find the probability of failure. The finite difference method (FDM) and the complex perturbation method (CPM) Kiran et al. (2017) compute gradients by repeated analyses with perturbed parameter values. Although the CPM is more accurate than the FDM, it requires complex arithmetic to be carried out through all levels of a finite element software framework. The adjoint system method (ASM) and the direct differentiation method (DDM) compute gradients analytically. The ASM uses Lagrange multipliers and is limited to specific types of structural response, mainly it cannot be used for path-dependent behavior.

The DDM implements analytic derivatives of the finite element response alongside the regular response implementation Kleiber et al. (1997). At the one-time expense of derivation, implementation, and debugging, the DDM gives accurate and efficient derivatives of the structural response. The DDM has been applied at all levels of nonlinear structural finite element analysis ranging from the material and section levels to the element formulations for material and geometric nonlinearity inside the basic system Scott et al. (2004). In addition, the DDM has been developed for the corotational formulation of geometric nonlinearity outside the basic system for two-dimensional problems Scott and Filippou (2007), but has not been developed for the corotational transformation in three dimensions due to the complexity of rotational transformations in space.

The objective of this paper is to apply the DDM to the corotational geometric transformation for frame finite elements in three dimensions. Analytic derivatives are obtained for the nodal triads and quaternions that define the orientation and deformation of a frame element in three-dimensional space. A verification example shows that the DDM corotational response sensitivity is correct by comparison with finite difference computations of the derivatives. The DDM equations developed herein have been implemented in the OpenSees finite element software framework.

2. COROTATIONAL FORMULATION IN THREE-DIMENSIONS

Three-dimensional frame elements have 12 degrees-of-freedom (DOFs), as shown in Figure 1, collected in four vectors of nodal displacements and rotations at each end of the element

$$\mathbf{u}^T = \begin{bmatrix} \mathbf{u}_I^T & \gamma_I^T & \mathbf{u}_J^T & \gamma_J^T \end{bmatrix} \tag{1}$$

where the translations at ends I and J are

$$\mathbf{u}_I = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \quad \mathbf{u}_J = \begin{bmatrix} u_7 \\ u_8 \\ u_9 \end{bmatrix} \tag{2}$$

while the rotations at ends I and J are

$$\gamma_I = \begin{bmatrix} u_4 \\ u_5 \\ u_6 \end{bmatrix} \quad \gamma_J = \begin{bmatrix} u_{10} \\ u_{11} \\ u_{12} \end{bmatrix} \tag{3}$$

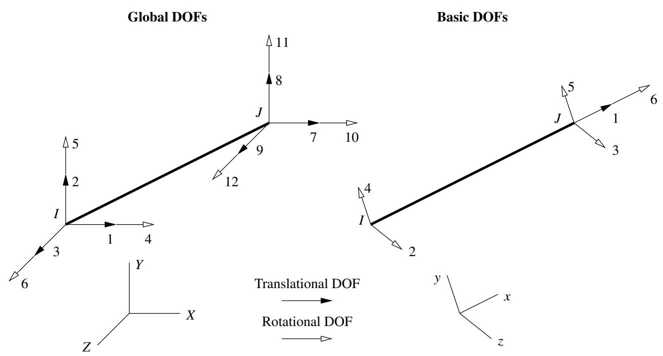

Figure 1: Degrees of freedom of global and basic coordinate systems of space frame (3D) element.

Six rigid body displacement modes correspond to these DOFs, leaving six basic DOFs that measure the element deformations

$$\mathbf{u}_b = \begin{bmatrix} u_{b1} \\ u_{b2} \\ u_{b3} \\ u_{b4} \\ u_{b5} \\ u_{b6} \end{bmatrix} = \begin{bmatrix} L_n - L \\ \theta_{I3} \\ \theta_{J3} \\ -\theta_{I2} \\ -\theta_{J2} \\ \theta_{J1} - \theta_{I1} \end{bmatrix} \tag{4}$$

where L_n is the deformed element length, which is computed from the nodal coordinate offset, $\mathbf{X}_{IJ} = \mathbf{X}_J - \mathbf{X}_I$, and relative nodal translations, $\mathbf{u}_{IJ} = \mathbf{u}_J - \mathbf{u}_I$

$$L_n = \|\mathbf{X}_{IJ} + \mathbf{u}_{IJ}\|_2 \tag{5}$$

The rotations θ_{I1} through θ_{J3} in Equation (4) are measured with respect to the *local* element axes, a coordinate system in between the global and basic systems. Due to the non-vectorial nature of rotations in three-dimensions, an approximate algorithm is required in order to compute the six basic deformations from the nodal rotations at the element ends.

To this end, a normalized unit quaternion Wehage (1984); Spring (1986) is defined for the nodal rotations at each of the element

$$\mathbf{q} = \begin{bmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{bmatrix} = \begin{bmatrix} \cos(\theta/2) \\ \sin(\theta/2)\mathbf{t} \end{bmatrix} \tag{6}$$

where θ and $\mathbf{t}$ are the magnitude and unit vector, respectively, of the nodal rotations at either end of the element. The matrix that describes the rotation of a point in space to a second point is described in terms of the unit quaternion

$$\mathbf{R} = \begin{bmatrix} q_0^2+q_1^2-0.5 & q_1q_2-q_0q_3 & q_1q_3+q_0q_2 \\ q_2q_1+q_0q_3 & q_0^2+q_2^2-0.5 & q_2q_3-q_1q_0 \\ q_3q_1-q_0q_2 & q_3q_2+q_0q_1 & q_0^2+q_3^2-0.5 \end{bmatrix} \tag{7}$$

Full details on the quaternion and the computation of the rotation matrix are omitted here, but can be found in Crisfield (1997); De Souza (2000).

Three triads define the orientation of an element in its displaced configuration. The element base triad, $\mathbf{E}$, describes the orientation of the element, while the nodal triads $\mathbf{N}_I$ and $\mathbf{N}_J$ describe the rotations at ends I and J, respectively.

The base vector, $\mathbf{e}_1$, defines the element chord, the line from end I to end J, in the displaced configuration

$$\mathbf{e}_1 = \frac{\mathbf{X}_{IJ} + \mathbf{u}_{IJ}}{L_n} \tag{8}$$

The other two vectors, $\mathbf{e}_2$ and $\mathbf{e}_3$, in the element base triad depend on the nodal rotations in the deformed configuration. The Crisfield algorithm Crisfield (1997) is a common approach to obtain these vectors in terms of the mean triad matrix

$$\bar{\mathbf{R}} = \begin{bmatrix} \bar{\mathbf{r}}_1 & \bar{\mathbf{r}}_2 & \bar{\mathbf{r}}_3 \end{bmatrix} = \mathbf{R}\left(\frac{\zeta}{2}\right)\mathbf{N}_I \tag{9}$$

where ζ is the pseudo-rotation vector De Souza (2000). Then, the basis vectors are expressed in terms of the mean triad and the base vector of the element chord

$$\mathbf{e}_2 = \bar{\mathbf{r}}_2 - \frac{\bar{\mathbf{r}}_2^T\mathbf{e}_1}{2}(\bar{\mathbf{r}}_1 + \mathbf{e}_1) \tag{10}$$

and

$$\mathbf{e}_3 = \bar{\mathbf{r}}_3 - \frac{\bar{\mathbf{r}}_3^T\mathbf{e}_1}{2}(\bar{\mathbf{r}}_1 + \mathbf{e}_1) \tag{11}$$

3. DIRECT DIFFERENTIATION OF THE COROTATIONAL FORMULATION

Direct differentiation of the 3D corotational transformation begins with the sensitivity of the unit base vectors in the displaced configuration of an element. The derivative of the unit base vector, $\mathbf{e}_1$, with respect to an uncertain parameter, h, of the structural model is found from Equation (8)

$$\frac{\partial \mathbf{e}_1}{\partial h} = \frac{L_n\left(\frac{\partial \mathbf{X}_{IJ}}{\partial h} + \frac{\partial \mathbf{u}_{IJ}}{\partial h}\right) - (\mathbf{X}_{IJ} + \mathbf{u}_{IJ})\frac{\partial L_n}{\partial h}}{L_n^2} \tag{12}$$

where $\partial \mathbf{X}_{IJ}/\partial h$ is non-zero only if h corresponds to a nodal coordinate at the element ends and $\partial \mathbf{u}_{IJ}/\partial h$ comes from the nodal displacement sensitivity. The derivative of the deformed element length, L_n, defined in Equation (5) is

$$\frac{\partial L_n}{\partial h} = \frac{1}{L_n}(\mathbf{X}_{IJ} + \mathbf{u}_{IJ})^T\left(\frac{\partial \mathbf{X}_{IJ}}{\partial h} + \frac{\partial \mathbf{u}_{IJ}}{\partial h}\right) \tag{13}$$

Likewise, the derivative of the base vectors, $\mathbf{e}_2$ and $\mathbf{e}_3$, are obtained from differentiating Equation (10) with respect to h

$$\begin{aligned}\frac{\partial \mathbf{e}_2}{\partial h} &= \frac{\partial \bar{\mathbf{r}}_2}{\partial h} - \frac{1}{2}\left[\left(\frac{\partial \bar{\mathbf{r}}_2^T}{\partial h}\mathbf{e}_1 + \bar{\mathbf{r}}_2^T\frac{\partial \mathbf{e}_1}{\partial h}\right)(\bar{\mathbf{r}}_1 + \mathbf{e}_1)\right. \\ &\quad \left. + \bar{\mathbf{r}}_2^T\mathbf{e}_1\left(\frac{\partial \bar{\mathbf{r}}_1}{\partial h} + \frac{\partial \mathbf{e}_1}{\partial h}\right)\right]\end{aligned} \tag{14}$$

and similarly Equation (11)

$$\begin{aligned}\frac{\partial \mathbf{e}_3}{\partial h} &= \frac{\partial \bar{\mathbf{r}}_3}{\partial h} - \frac{1}{2}\left[\left(\frac{\partial \bar{\mathbf{r}}_3^T}{\partial h}\mathbf{e}_1 + \bar{\mathbf{r}}_3^T\frac{\partial \mathbf{e}_1}{\partial h}\right)(\bar{\mathbf{r}}_1 + \mathbf{e}_1)\right. \\ &\quad \left. + \bar{\mathbf{r}}_3^T\mathbf{e}_1\left(\frac{\partial \bar{\mathbf{r}}_1}{\partial h} + \frac{\partial \mathbf{e}_1}{\partial h}\right)\right]\end{aligned} \tag{15}$$

The remaining unknowns in the two preceding equations are the derivatives of the mean rotation triad. This derivative is obtained from Equation (9)

$$\begin{aligned}\frac{\partial \bar{\mathbf{R}}}{\partial h} &= \left[\begin{array}{ccc} \frac{\partial \bar{\mathbf{r}}_1}{\partial h} & \frac{\partial \bar{\mathbf{r}}_2}{\partial h} & \frac{\partial \bar{\mathbf{r}}_3}{\partial h}\end{array}\right] \\ &= \mathbf{R}\left(\frac{\zeta}{2}\right)\frac{\partial \mathbf{N}_I}{\partial h} + \frac{\partial \mathbf{R}\left(\frac{\zeta}{2}\right)}{\partial h}\mathbf{N}_I \end{aligned} \quad (16)$$

where the derivative of the rotation matrix is obtained from differentiation of Equation (7), which in turn depends on the derivative of the unit quaternion defined in Equation (6)

$$\frac{\partial \mathbf{q}}{\partial h} = \left[\begin{array}{c} -0.5\sin(\theta/2)\frac{\partial \theta}{\partial h} \\ \frac{\sin(\theta/2)}{\theta}\frac{\partial \theta}{\partial h} + \left(\frac{\theta}{2}\cos(\theta/2) - \sin(\theta/2)\right)\frac{\partial \theta}{\partial h}\mathbf{t}/\theta \end{array}\right] \quad (17)$$

Additional details on the derivative of the quaternion and the rotation matrix are found in Al-Aukaily (2017).

3.1. *Sensitivity of the local-basic transformation*

With the derivative of the element orientation defined, attention now turns to the sensitivity of the equilibrium and compatibility relationships between the local and basic coordinate systems. The displacements transform from the local system to deformations of the basic system via

$$\mathbf{u}_b = \mathbf{T}_{bl}\mathbf{u}_l \quad (18)$$

where $\mathbf{T}_{bl}$ is a matrix of ones and zeros. Accordingly, the derivative of this compatibility relationship is

$$\frac{\partial \mathbf{u}_b}{\partial h} = \mathbf{T}_{bl}\frac{\partial \mathbf{u}_l}{\partial h} \quad (19)$$

Assuming there are no member loads, the conjugate equilibrium relationship is given by the transpose of the compatibility matrix

$$\mathbf{p}_l = \mathbf{T}_{bl}^T\mathbf{p}_b \quad (20)$$

and the derivative of the equilibrium relationship is

$$\left.\frac{\partial \mathbf{p}_l}{\partial h}\right|_{\mathbf{u}_l} = \mathbf{T}_{bl}^T\left.\frac{\partial \mathbf{p}_b}{\partial h}\right|_{\mathbf{u}_b} \quad (21)$$

3.2. *Sensitivity of the global-local transformation*

The transformation from displacements in the global coordinate directions to the local element displacements is given by

$$\mathbf{u}_l = \mathbf{T}\mathbf{u} \quad (22)$$

where $\mathbf{T}$ is defined in terms of the element basis vectors and mean triads

$$\mathbf{T}^T = \left[\begin{array}{ccccccc} \mathbf{t}_1^T & \mathbf{t}_2^T & \mathbf{t}_3^T & \mathbf{t}_4^T & \mathbf{t}_5^T & \mathbf{t}_{6I}^T & \mathbf{t}_{6J}^T \end{array}\right] \quad (23)$$

where $\mathbf{t}_1$ corresponds to axial deformation

$$\mathbf{t}_1 = \left[\begin{array}{cccc} -\mathbf{e}_1^T & \mathbf{0}^T & \mathbf{e}_1^T & \mathbf{0} \end{array}\right] \quad (24)$$

and $\mathbf{t}_2$ corresponds to flexural deformation

$$\mathbf{t}_2 = \frac{1}{2\cos(\theta_{I3})}\left[\mathbf{L}(\bar{\mathbf{r}}_2)\mathbf{n}_{I1} + \mathbf{h}_{I3}\right]^T \quad (25)$$

with $\mathbf{L}$ and $\mathbf{h}$ defined in Al-Aukaily (2017) and similar definitions for $\mathbf{t}_3$ through $\mathbf{t}_{6J}$. The derivative of the compatibility relationship is then

$$\frac{\partial \mathbf{u}_l}{\partial h} = \mathbf{T}\frac{\partial \mathbf{u}}{\partial h} + \frac{\partial \mathbf{T}}{\partial h}\mathbf{u} \quad (26)$$

where

$$\frac{\partial \mathbf{T}^T}{\partial h} = \left[\begin{array}{ccccccc} \frac{\partial \mathbf{t}_1^T}{\partial h} & \frac{\partial \mathbf{t}_2^T}{\partial h} & \frac{\partial \mathbf{t}_3^T}{\partial h} & \frac{\partial \mathbf{t}_4^T}{\partial h} & \frac{\partial \mathbf{t}_5^T}{\partial h} & \frac{\partial \mathbf{t}_{6I}^T}{\partial h} & \frac{\partial \mathbf{t}_{6J}^T}{\partial h} \end{array}\right] \quad (27)$$

$$\frac{\partial \mathbf{t}_1}{\partial h} = \left[\begin{array}{cccc} -\frac{\partial \mathbf{e}_1^T}{\partial h} & \mathbf{0}^T & \frac{\partial \mathbf{e}_1^T}{\partial h} & \mathbf{0}^T \end{array}\right] \quad (28)$$

and the derivatives of $\mathbf{t}_2$ through $\mathbf{t}_{6J}$ are defined in Al-Aukaily (2017).

Equilibrium of element forces between the local and global systems is

$$\mathbf{p} = \mathbf{T}^T\mathbf{p}_l \quad (29)$$

where the derivative of the equilibrium relationship is

$$\left.\frac{\partial \mathbf{p}}{\partial h}\right|_{\mathbf{u}} = \mathbf{T}^T\left(\mathbf{k}_l\frac{\partial \mathbf{T}}{\partial h}\mathbf{u} + \left.\frac{\partial \mathbf{p}_l}{\partial h}\right|_{\mathbf{u}_l}\right) + \frac{\partial \mathbf{T}^T}{\partial h}\mathbf{p}_l \quad (30)$$

This derivative of element forces in the global coordinate system is assembled in to the structural level equations of DDM response sensitivity and incorporated in a two-phase computation process for path-dependent constitutive models Zhang and Der Kiureghian (1993).

4. NUMERICAL EXAMPLE

A frame structure demonstrates that the DDM equations for the three-dimensional corotational transformation are correct. Verification of the DDM takes place against finite difference computations for uncertain material, cross-section, and nodal coordinate parameters of the frame. This frame has been studied by several researchers in the development of large displacement frame finite element formulations Argyris (1982); Abbasnia and Kassimali (1995); De Souza (2000).

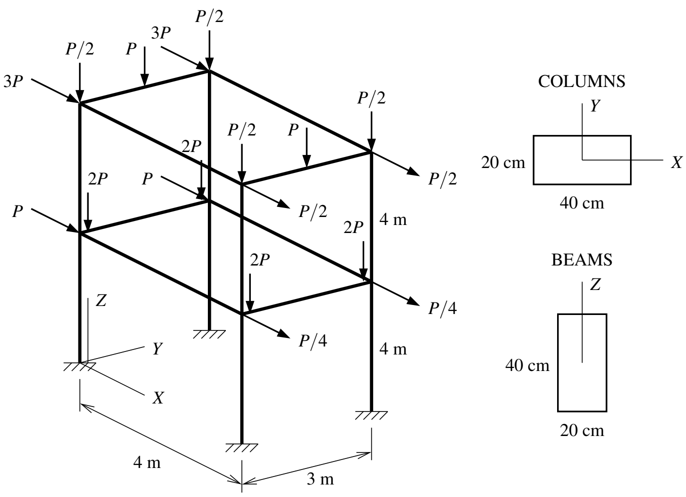

Figure 2: Two-story frame with cross-section dimensions and pattern of applied loads.

The frame dimensions and loading are shown in Figure 2. The stress-strain response of the frame members is assumed to be elastic-perfectly-plastic (EPP) with modulus E=19613 MPa and yield strength f_y=98 MPa. All members are oriented for strong-axis bending for the direction of lateral load shown in Figure 2. The response of each member is simulated using one material nonlinear force-based frame element Neuenhofer and Filippou (1997) with four Gauss-Lobatto integration points and fiber-discretized cross-sections.

The load-displacement response of the frame is shown in Figure 3 with the reference load value P and the roof displacement, U. It is noted that all loads, vertical and lateral, increase linearly with respect to pseudo-time in the nonlinear analysis using displacement control static integration Clarke and Hancock (1990). After reaching a peak load value of $P \approx 128$ kN at a roof displacement of $U \approx 70$ cm, the frame starts to lose load carrying due to both material and geometric nonlinearity.

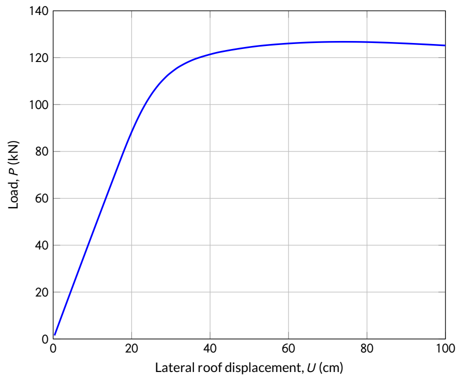

Figure 3: Load-displacement relationship for two-story frame.

The DDM sensitivity of the frame response is computed with respect to the yield strength, f_y, of the column members. Comparisons of the DDM response sensitivity with finite difference computations using a parameter perturbation of $0.0001 f_y$ are shown in Figure 4 for the load factor, λ, and the roof displacement, U, in the displacement-controlled pushover analysis Al-Aukaily and Scott (2018).

In addition to verification of the DDM by matching finite difference computations, Figure 4 shows the onset of material yielding in the structure at a roof displacement of $U \approx 20$ cm, where the sensitivity with respect to f_y becomes non-zero for both the load factor and the roof displacement. The positive values for sensitivity of the load factor Figure 4 (a)) indicate that as f_y increases, the load factor must also increase in order to reach the same target displacement in the displacement-controlled analysis. On the other hand, the negative sensitivity values for displacement (Figure 4 (b)) indicate that as f_y increases, the roof displacement will decrease.

Sharp transitions in the displacement response sensitivity are noted in Figure 4 (b). These transitions are due to the yielding of fibers in the beam and column member cross-sections and they are observed in both the DDM and finite difference sensitivity response indicating that the transitions are not an artifact of either numerical method, but rather an inherent behavior of the underlying finite element model.

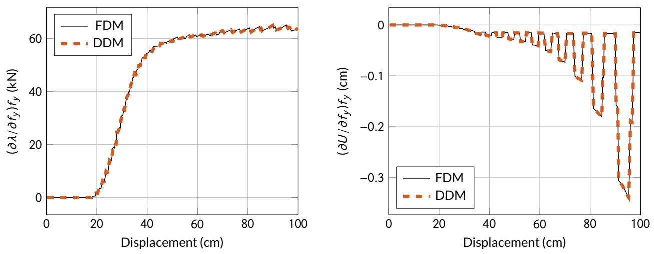

Figure 4: Response sensitivity of load factor and lateral roof displacement with respect to the yield strength, f_y, of all members.

5. CONCLUSIONS

The direct differentiation method (DDM) provides accurate gradients of the large displacement response with respect to uncertain parameters of three-dimensional frame finite elements that utilize the corotational transformation. Although analytical derivatives of the corotational transformation are complex, their implementation in the OpenSees finite element software framework leads to efficient gradient computations compared to perturbation-based computations, e.g., finite differences. The verified DDM implementation for corotational space frame finite elements supports a wide range of gradient-based applications such as reliability, optimization, and system identification for complex structures.

6. REFERENCES

Abbasnia, R. and Kassimali, A. (1995). "Large deformation elastic-plastic analysis of space frames." *Journal of Constructional Steel Research*, 35(3), 275–290.

Al-Aukaily, A. (2017). "Response Sensitivity Formulations for Geometrically Nonlinear Finite Element Analysis." Ph.D. thesis, Oregon State University, Oregon State University.

Al-Aukaily, A. and Scott, M. H. (2018). "Sensitivity analysis for displacement-controlled finite-element analyses." *Journal of Structural Engineering*, 144(3), 04017222.

Argyris, J. (1982). "An excursion into large rotations." *Computer Methods in Applied Mechanics and Engineering*, 32(1), 85–155.

Clarke, M. and Hancock, G. (1990). "A study of incremental-iterative strategies for non-linear analysis." *Internaitional Journal for Numerical Methods in Engineering*, 29(7), 1365–1391.

Crisfield, M. A. (1997). *Non-linear Finite Element Analysis of Solids and Structures, Volume 2*. Wiley, Chichester, UK.

De Souza, R. M. (2000). "Force-based Finite Element for Large Displacement Inelastic Analysis of Frames." Ph.D. thesis, University of California, Berkeley, University of California, Berkeley.

Kiran, R., Li, L., and Khandelwal, K. (2017). "Complex perturbation method for sensitivity analysis of nonlinear trusses." *Journal of Structural Engineering*, 143(1), 04016154.

Kleiber, M., Antunez, H., Hien, T., and Kowalczyk, P. (1997). *Parameter Sensitivity in Nonlinear Mechanics*. Wiley, New York, NY.

Neuenhofer, A. and Filippou, F. (1997). "Evaluation of nonlinear frame finite element models." *Journal of Structural Engineering*, 123(7), 958–966.

Scott, M. H. and Filippou, F. C. (2007). "Response gradients for nonlinear beam-column elements under large displacements." *Journal of Structural Engineering*, 133(2), 155–165.

Scott, M. H., Franchin, P., Fenves, G. L., and Filippou, F. C. (2004). "Response sensitivity for nonlinear beam-column elements." *Journal of Structural Engineering*, 130(9), 1281–1288.

Spring, K. W. (1986). "Euler parameters and the use of quaternion algebra in the manipulation of finite rotations: A review." *Mechanism and Machine Theory*, 21(5), 365–373.

Wehage, R. A. (1984). "Quaternions and Euler parameters – A brief exposition." *Computer Aided Analysis and Optimization of Mechanical System Dynamics*, E. J. Haug, ed., Springer, 147–180.

Zhang, Y. and Der Kiureghian, A. (1993). "Dynamic response sensitivity of inelastic structures." *Computer Methods in Applied Mechanics and Engineering*, 108(1-2), 23–36.